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Ultimate Stochastic Entities

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We take the view that everything that is known about a physical system can be described by a "stochastic entity" $(\mathscr{A}, \Delta)$, which consists of a "manual" $\mathscr{A}$ of experiments that can be performed on the system, and a set Δ of possible stochastic states (probability measures) on the logic of the manual. We next consider what happens when new information about the system is learned and describe precisely how one then obtains a new stochastic entity more elaborate than the first. Finally, we show that as information about the system continues to grow, the increasingly elaborate stochastic entities describing the system necessarily acquire mathematical properties often assumed for mathematical convenience in papers on quantum mechanics.

1. INTRODUCTION

Foulis and Randall have introduced the manual as a basic structure in the operational approach to quantum mechanics (Foulis *et al.*, 1983; Foulis and Randall, 1981a,b, 1978). One realization of a manual is as a prescribed set of experiments, which may be performed to obtain information about a physical system. Given one manual, one might wish for a more elaborate manual to learn more about the system. In this paper we are concerned with describing an order relation which reflects an order of knowledge about a physical system.

Our basic structure is a stochastic entity $(\mathscr{A}, \Delta)$ consisting of a manual $\mathscr{A}$ and a nonempty convex set of states $\Delta \subset \Omega(\Pi(\mathscr{A}))$ (Randall and Foulis, 1983), where $\Pi(\mathscr{A})$ is the operational logic of $\mathscr{A}$ (Foulis *et al.*, 1983). We define a partial ordering on a set $\mathfrak{P}$ of stochastic entities and define an "ultimate" stochastic entity as a limit of an inductive subset of $\mathfrak{P}$ in a manner suggested by Fischer and Rüttimann (1978a). The careful choice this subset is based on the method of forcing discovered by Paul Cohen (1966).

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Ultimate stochastic entities represent "ultimate knowledge" about physical systems. Such entities satisfy many properties and give new impetus to investigations that had taken these properties as *ad hoc* assumptions. In a paper to follow we shall connect these stochastic entities to physical entities in the sense of Foulis *et al.* (1983).

2. PRELIMINARIES

We shall assume that the reader is familiar with the notions of a manual $\mathscr{A}$ and its operational logic $\Pi(\mathscr{A})$ as they are defined in Foulis *et al.* (1983). Throughout this paper we assume that every manual $\mathscr{A}$ is orthocoherent, so that $\Pi(\mathscr{A})$, together with the usual order relation and orthocomplementation, is an orthomodular poset (Rüttimann, 1984). In the sequel we shall make use of the following, which are explained in Rüttimann (to appear) with more detail than we shall supply here.

If $(P, \leq, ')$ is any orthomodular poset, then we refer to a probability measure on P as a *stochastic state* on P, and we denote by $\Omega(P)$ the set of all stochastic states on P. Let $(P, \leq, ')$ be an orthomodular poset and Δ a nonempty, convex subset of $\Omega(P)$. We denote by $W(P)$ the linear subspace of R^P (R = the reals) consisting of all measures on P, and we denote $V(\Delta) := \mathrm{lin}(\Delta) \subset W(P)$. We organize $V(\Delta)$ into a base-norm space with Δ as a base for the generating cone. We denote by $V(\Delta)^*$ the Banach dual of base-norm space $V(\Delta)$ and organize $V(\Delta)^*$ into an order unit norm space. We denote the order unit by $1(\Delta)$, and the order unit interval by $[0, 1(\Delta)]$. For $q \in P$, a map $e(\Delta)(q)$: $V(\Delta) \to R$ is defined by $e(\Delta)(q)(\mu) := \mu(q)$, and $e(\Delta): P \to [0, 1(\Delta)]$ is called the *evaluation map.* For $q \in P$, we denote $f_q := e(\Delta)(q)$.

If $\mathscr{A}$ and $\mathscr{B}$ are manuals, then a *morphism* from $\mathscr{A}$ to $\mathscr{B}$ is a map $\varphi: \bigcup\mathscr{A} \to \bigcup\mathscr{B}$ such that if $E \in \mathscr{A}$, then $\varphi(E) := \bigcup\{\varphi(x) | x \in E\} \in \mathscr{B}$. A morphism φ is called *strict* if for all $x, y \in \bigcup\mathscr{A}$, $x \perp y$ if and only if $\varphi(x) \perp \varphi(y)$. We denote by $\mathrm{Mor}_0(\mathscr{A}, \mathscr{B})$ the set of strict morphisms from $\mathscr{A}$ to $\mathscr{B}$. Note that the terminology we are using here is that of Fischer and Rüttimann (1978a). What we are calling "strict morphisms" are called "outcome-preserving, operation-preserving, faithful conditionings" in Foulis and Randall (1978), where morphisms between manuals were first introduced in detail.

Strict morphisms can be "lifted" to logics. Let $\mathscr{A}$, $\mathscr{B}$ be manuals and suppose $\varphi \in \mathrm{Mor}_0(\mathscr{A}, \mathscr{B})$. Denote by $\bar{\varphi}$ the map from $\Pi(\mathscr{A})$ to $\Pi(\mathscr{B})$ defined by $\bar{\varphi}(p(A)) = p(\varphi(A))$. It is not difficult to verify that $\bar{\varphi}$ is well defined. Furthermore, it can be verified without difficulty that $\bar{\varphi}$ preserves "op" (Foulis and Randall, 1983) and set inclusion, so that $\bar{\varphi}$ preserves the implication relation $\leq$ on manuals (Foulis and Randall, 1983). From this one shows easily that $\bar{\varphi}$ is an order-preserving map between operational logics. It can be shown from the strictness of φ that $\bar{\varphi}$ preserves orthocomplementation, hence orthogonality between operational logics.

Finally, we shall make use of the following abuses of notation. Suppose $\mathscr{A}$ is a manual and $\mu \in \Omega(\Pi(\mathscr{A}))$. For $x \in \bigcup \mathscr{A}$ we shall write $p(x)$ for $p(\{x\})$ and $\mu(x)$ for $\mu(p\{x\})$. We also write f_x (respectively f_A) for f_q (respectively f_r) when $q = p(x)$, $x \in \bigcup \mathscr{A}$ (respectively, $r = p(A)$, A an $\mathscr{A}$ event).

3. ULTIMATE STOCHASTIC ENTITIES

3.1. Definition. A *stochastic entity* is a pair $S = \langle \mathscr{M}(S), \Delta(S) \rangle$ consisting of a manual $\mathscr{M}(S)$ and a nonempty convex subset of states $\Delta(S) \subset \Omega(\Pi(\mathscr{M}(S)))$. We denote the collection of all stochastic entities by $\mathscr{S}\mathscr{E}$.

3.2. Definition. Suppose S, $T \in \mathscr{S}\mathscr{E}$. An $\mathscr{S}\mathscr{E}$ *morphism* from S to T is a pair of maps $\varphi = ({}_1\varphi, {}_2\varphi)$ satisfying (i) ${}_1\varphi \in \mathrm{Mor}_0(\mathscr{M}(S), \mathscr{M}(T))$ and ${}_2\varphi \colon \Delta(S) \to \Delta(T)$, (ii) If $x \in \bigcup \mathscr{M}(S)$ and $\mu \in \Delta(S)$; then $({}_2\varphi(\mu))({}_1\varphi(x)) = \mu(x)$.

We denote by $\mathscr{S}\mathscr{E}$-Mor(S, T) the set of all $\mathscr{S}\mathscr{E}$ morphisms from S to T.

If φ is an $\mathscr{S}\mathscr{E}$ morphism from S to T, we interpret $\langle \mathscr{M}(T), \Delta(T) \rangle$ as a stochastic entity that embodies all the information available from S and perhaps more. We think of $\mathscr{M}(T)$ as a laboratory manual more elaborate than $\mathscr{M}(S)$, containing the instructions for every operation in $\mathscr{M}(S)$ as well as new operations. Every state in $\Delta(S)$ has an extension to a state in $\Delta(T)$. We shall refer to T as an *elaboration* of S. Later we shall consider the case that ${}_2\varphi$ is a surjection, reflecting the situation that the set of states of a system does not increase merely because we find new experiments to test for those states. For now, however, we allow that $\Delta(T)$ contains states of which one is unaware when one is considering the less elaborate stochastic entity S.

Next we introduce partially ordered systems of stochastic entities.

3.3. Definition. A pair $\langle X, F \rangle$ is an *inductive system of stochastic entities* if and only if the following are true:

(i) X is a set of stochastic entities and F is a set of $\mathscr{S}\mathscr{E}$ morphisms.

(ii) If S, $T \in X$, then there exists at most one $\mathscr{S}\mathscr{E}$ morphism $\varphi_{S,T} \in F$ from S to T.

(iii) The relation $\leq$ on X defined by

$$S \leq T \Leftrightarrow \exists \varphi_{S,T} \in F \cap \mathscr{S}\mathscr{E}\ \mathrm{Mor}(S, T)$$

is a partial ordering with respect to which X is a directed set.

(iv) If $S \in X$ and $\varphi_{S,S} \in F$, then ${}_1\varphi_{S,S}$ and ${}_2\varphi_{S,S}$ are identity maps.

We now consider an inductive system $\langle X, F\rangle$ and show how one can construct a stochastic entity that is a limit of $\langle X, F\rangle$. We begin with the construction suggested in Fischer and Rüttimann (1978a).

Define $X^{\wedge} := \bigcup\{\mathcal{M}(T) | T \in X\}$ and an equivalence relation on $X^{\wedge}$ as follows: if $x, y \in X^{\wedge}$, then $x \sim y$ if and only if there exists $\varphi, \psi \in F$ with ${}_1\varphi(x) = {}_1\psi(y)$. Now for each $x \in X^{\wedge}$, denote by $x\ddagger$ the equivalence class determined by x and for each $\mathcal{M}(T)$-event A, $T \in X$, let $A\ddagger := \{x\ddagger | x \in A\}$. Finally, define $X_0 := \{E\ddagger | E \in \mathcal{M}(T)$ for some $T \in X\}$. Note that $\bigcup X_0 = X^{\wedge}/\sim$. It is not difficult to verify that X_0 is a manual.

For $T \in X$, we claim that the map $x \to x\ddagger$, $x \in \bigcup\mathcal{M}(T)$ is a strict morphism from $\mathcal{M}(T)$ to X_0. To see that it is strict, first suppose that $x, y \in \bigcup\mathcal{M}(T)$, $x \perp y$. Then $\exists E \in \mathcal{M}(T)$ with $x, y \in E$. Then $x\ddagger, y\ddagger \in E\ddagger$. We will have that $x\ddagger \perp y\ddagger$ if we can show that they are unequal. But if $x\ddagger = y\ddagger$, then $\exists \varphi \in F$ with ${}_1\varphi(x) = {}_1\varphi(y)$, a contradiction of the fact that ${}_1\varphi$ is strict. Conversely, suppose $x\ddagger \perp y\ddagger$, $x\, y \in \bigcup\mathcal{M}(T)$. Then $\exists R \in X$, $R \geq T$ and $\mathcal{M}(R)$ event A with $x\ddagger, y\ddagger \in A\ddagger$, and ${}_1\varphi_{T,R}(x), {}_1\varphi_{T,R}(y) \in A$, ${}_1\varphi_{T,R}(x) \neq {}_1\varphi_{T,R}(y)$. Since ${}_1\varphi_{T,R}$ is strict, we conclude that $x \perp y$.

For $T \in X$, the map $x \to x\ddagger$ can be lifted to a map from $\Pi(\mathcal{M}(T))$ to $\Pi(X_0)$. For $q \in \Pi(\mathcal{M}(T))$, we write $q\ddagger$ for the image of q under the lifted map.

Now let $\Delta_1 := \bigcup\{\Delta(S) | S \in X\}$, and define an equivalence relation on Δ_1 as follows: if $S, T \in X$ and $\mu \in \Delta(S)$, $\nu \in \Delta(T)$, then $\mu \equiv \nu$ if and only if $\exists \varphi, \psi \in F$ with ${}_2\varphi(\mu) = {}_2\psi(\nu)$. Finally, define $\Delta_0 := \Delta_1/\equiv$. We denote by $\mu\ddagger$ the equivalence class determined by $\mu \in \Delta_1$.

We now claim that the members of Δ_0 are states on $\Pi(X_0)$. Suppose that $S, T \in X$ and $\mu \in \Delta(S)$ and $q = p(A) \in \Pi(\mathcal{M}(T))$. Since X is directed, $\exists R \in X$ with $S, T \leq R$. We define $\mu\ddagger(q\ddagger) := ({}_2\varphi_{S,R}(\mu))({}_1\bar{\varphi}_{T,R}(q))$. By straightforward arguments using the fact that X is a directed set one can show that $\mu\ddagger(q\ddagger)$ is well defined and that $\mu\ddagger$ is indeed a state on $\Pi(X_0)$.

We thus have the following:

3.4. Theorem. If $\langle X, F\rangle$ is an inductive system of stochastic entities, then $\langle X_0, \Delta_0\rangle$ is a stochastic entity.

In accord with our notation, we shall write $\mathcal{S}\langle X, F\rangle$ for the stochastic entity with $\mathcal{M}(\mathcal{S}\langle X, F\rangle) = X_0$ and $\Delta(\mathcal{S}\langle X, F\rangle) = \Delta_0$.

Now we begin construction of an "ultimate stochastic entity." Let S be a stochastic entity and define $\mathfrak{P}(S) := \{\langle Y, H\rangle | \langle Y, H\rangle$ is an inductive system of stochastic entities with Y finite and $S \in Y$.$\}$ We write $\mathfrak{P}$ for $\mathfrak{P}(S)$ to simplify notation, and we call $\mathfrak{P}$ *the finite elaboration system for S.*

As it is defined here $\mathfrak{P}(S)$ is not a set but a proper class. Strictly speaking, therefore, we are working in a system such as Von Neumann–Bernays–Godel set theory (Drake, 1974). It would be possible to remain in Zermelo–Fraenkel set theory, limiting the size of $\mathfrak{P}(S)$ by bounding the set-theoretic rank of its elements, for example, but that is not necessary for our purpose.

Next we define a partial order on $\mathfrak{P}$.

3.5. Definition. $\langle Y, H\rangle \leq \langle Y', H'\rangle$ if and only if (i) $Y \subset Y'$ and $H \subset H'$, (ii) If $R, T \in Y$ with $R \leq T$, then $\varphi_{R,T} \notin H' \backslash H$.

If S is a stochastic entity consisting of a manual of experiments and a set of states representing a physical system, then we think of an element of $\mathfrak{P}(S)$ as a directed network of stochastic entities, each investigating the same system but in a variety of ways. For example, S might be the stochastic entity used by a particular research group R at a particular time t. Then for $\langle Y, H\rangle \in \mathfrak{P}(S)$, Y might consist of stochastic entities for the investigation of the same system at different research groups and/or at different times. Some members of Y would be elaborations of S, perhaps resulting from technological developments unavailable to group R at time t. If $\langle Y', H'\rangle \geq \langle Y, H\rangle$, then Y' might extend Y in several ways: it could add elaborations resulting from synthesizing levels of expertise and technological developments at several laboratories, or it might contain hypothetical levels of development in the future. It might also represent refinements in the model of the physical system. In this case, we are thinking of morphisms as *interpretations*, which is the way they were first introduced in Foulis and Randall (1981a).

3.6. Definition. A set $\mathfrak{D} \subset \mathfrak{P}$ is called *dense* in $\mathfrak{P}$ if for every $\langle Y, H\rangle \in \mathfrak{P}$, $\exists \langle Y', H'\rangle \in \mathfrak{D}$ with $\langle Y, H\rangle \leq \langle Y', H'\rangle$.

3.7. Definition. A set $\mathfrak{G} \subset \mathfrak{P}$ is called *generic* if (i) $\mathfrak{G}$ is a directed set under $\leq$ and (ii) $\mathfrak{G} \cap \mathfrak{D} \neq \phi$ for every $\mathfrak{D}$ dense in $\mathfrak{G}$.

Suppose $\mathfrak{G} \subset \mathfrak{P}$ is generic. Let $X = X(\mathfrak{G}) := \bigcup\{Y | \exists H \ni \langle Y, H\rangle \in \mathfrak{G}\}$ and let $F = F(\mathfrak{G}) := \bigcup\{H | \exists Y \ni \langle Y, H\rangle \in \mathfrak{G}\}$. We claim that $\langle X, F\rangle$ is an inductive system of stochastic entities. We shall verify 3.3(ii) and leave the rest of the verification of 3.3 to the reader. Supposd $\langle Y, H\rangle, \langle Y', H'\rangle \in \mathfrak{G}$ and $R \in Y$, $T \in Y'$. Suppose further that $\varphi \in H$ and $\psi \in H'$ with $\varphi, \psi \in \mathcal{SE}\text{-Mor}(R, T)$. Since $\mathfrak{G}$ is directed, $\exists \langle Z, K\rangle \in \mathfrak{G}$ with $\langle Y, H\rangle, \langle Y', H'\rangle \leq \langle Z, K\rangle$. Then $R, T \in Z$ and $\varphi, \psi \in K$. So $\varphi = \psi$.

Now suppose $\mathfrak{G} \subset \mathfrak{P}(S)$ is a generic set and $\langle X(\mathfrak{G}), F(\mathfrak{G})\rangle$ is the inductive system of stochastic entities obtained from $\mathfrak{G}$ as above. Then by Theorem 3.4 $\mathcal{S}(\mathfrak{G}) = \mathcal{S}\langle X(\mathfrak{G}), F(\mathfrak{G})\rangle$ is a stochastic entity, which we shall call *the ultimate stochastic entity based on S and obtained from* $\mathfrak{G}$.

If we consider $\mathfrak{P}(S)$ to represent all possible patterns of epistemological advances over S, then $X(\mathfrak{G})$ represents a complete, consistent history of our knowledge of the system under unvestigation by S. If $T_1, T_2 \in X(\mathfrak{G})$ are two elaborations of S which are not comparable, we might consider them as reflecting the technological capacities of two research groups which are not in communication. The existence in $X(\mathfrak{G})$ of T_3, $T_3 \geq T_1$, T_2, guaranteed by 3.7(i), represents a situation in which the knowledge of the two groups has been synthesized.

Questions about the system under investigation by S are represented by dense sets. The elements of a dense set $\mathfrak{D}$ are those stochastic entities which are capable of answering the question. If the question is, for example, "Will outcomes x and y (not in S) be orthogonal?," then $\mathfrak{D}$ would be all stochastic entities in which both x and y appear as outcomes. The property 3.7(ii) of $\mathfrak{G}$ reflects our optimism that all such questions will be answered eventually.

The question of the existence of a generic set $\mathfrak{G}$ naturally arises at this point. We shall remark briefly on this question and refer the reader to Cohen (1966) and Kunen (1980) for a more complete discussion. It is easy to show that if we carry out our mathematics inside a model $\mathfrak{M}$ of set theory, no generic set exists in $\mathfrak{M}$. In the theory of "forcing," however, the machinery of Paul Cohen is used to construct a $\mathfrak{G}$ in a larger model (call it $\mathfrak{M}[\mathfrak{G}]$) such that $\mathfrak{G}$ is "generic over $\mathfrak{M}$," that is, $\mathfrak{G}$ meets all dense sets in $\mathfrak{M}$, (though not all dense sets in $\mathfrak{M}[\mathfrak{G}]$). The ultimate stochastic entity $\mathcal{S}(\mathfrak{G})$ exists in $\mathfrak{M}[\mathfrak{G}]$. This is just what we would expect. If indeed $\mathfrak{G}$ and $\mathcal{S}(\mathfrak{G})$ represent the complete history of the world, then they cannot exist in it. It is reasonable, however, that there are larger worlds capable of containing both our world and its history.

The ultimate stochastic entity depends on $\mathfrak{G}$. The differences in such entities reflect the ways in which $\mathfrak{G}$ "answers" the questions posed by the dense sets. In general, such variations are local, and the global properties we examine in the next section hold for all ultimate stochastic entities.

Our use of forcing is strongly reminiscent of Kripke's work in constructing models for intuitionistic mathematics. See, for example, Fitting (1969).

4. PROPERTIES OF ULTIMATE STOCHASTIC ENTITIES

One great advantage of generic sets is that while they exist only outside our model $\mathfrak{M}$, it is still possible to reason about them successfully inside $\mathfrak{G}$.

In this section we exhibit the properties of ultimate stochastic entities that establish them as natural objects to carry "ultimate" knowledge.

We begin with the following:

4.1. Lemma. Let S be a stochastic entity. Then there exists a stochastic entity T and $\mathcal{SE}$ morphism $\varphi \in \mathcal{SE}\text{-Mor}(S, T)$ such that (i) ${}_2\varphi: \Delta(S) \to \Delta(T)$ is a bijection, and (ii) for every $g \in [0, 1(\Delta(T))]$ there exists $x \in \bigcup(\mathcal{M}(T))$ with $g = g_x$. That is, $g(\nu) = \nu(x)$ for all $\nu \in \dot{\Delta}(T)$.

Proof. Let $D=\{f\in[0,1(\Delta(S))]|\exists x\in\bigcup(\mathcal{M}(S))$ with $f=f_x\}$. Associate with each $f\in D$ a pair $E_f=\{x_f, y_f\}$ of distinct elements such that (i) $f\in D\Rightarrow E_f\cap\bigcup\mathcal{M}(S)=\phi$, (ii) for $f, g\in D$ with $f\neq g$, $E_f\cap E_g=\phi$.

Let $\mathcal{B}=\mathcal{M}(S)\cup\{E_f|f\in D\}$. Let ${}_1\varphi: \bigcup\mathcal{M}(S)\to\bigcup\mathcal{B}$ be the inclusion map. Clearly, $\mathcal{B}$ is a manual and ${}_1\varphi$ is a strict morphism from $\mathcal{M}(S)$ to $\mathcal{B}$.

Let us describe the logic $\Pi(\mathcal{B})$. Let ${}_1\bar{\varphi}$ be the lift of ${}_1\varphi$, so that ${}_1\bar{\varphi}$: $\Pi(\mathcal{M}(S))\to\Pi(\mathcal{B})$ is an orthomodular poset isomorphism. It is not difficult to see that for every element $q\in\Pi(\mathcal{B})$, either $q\in$ image $({}_1\bar{\varphi})$ or else $\exists f\in D$ with $q=p(x_f)$ or $q=p(y_f)$.

Define map ${}_2\varphi$: $\Delta(S)\to\Omega(\Pi(\mathcal{B}))$ as follows: for $\mu\in\Delta(S)$, $q\in\Pi(\mathcal{B})$,

$$
{}_2\varphi(\mu)(q)=\begin{cases}\mu(r) & \text{if } q={}_1\bar{\varphi}(r),\ r\in\Pi(\mathcal{M}(S))\\ f(\mu) & \text{if } q=p(x_f), f\in D\\ 1-f(\mu) & \text{if } q=p(y_f), f\in D\end{cases}
$$

Clearly, ${}_2\varphi$ is injective, and a straight forward computation shows that ${}_2\varphi$ preserves convex combinations. That is, if $0\leq t\leq 1$ and μ_1, $\mu_2\in\Delta(S)$, then ${}_2\varphi(t\mu_1+(1-t)\mu_2)=t{}_2\varphi(\mu_1)+(1-t){}_2\varphi(\mu_2)$. From this and the fact that $\Delta(S)$ is a cone base for $V(\Delta(S))$, one can show that ${}_2\varphi$ has a unique extension to a linear map on all of $V(\Delta(S))$ (Fischer and Rüttimann, 1978b). Also, the image of ${}_2\varphi$ is a convex subset of $\Omega(\Pi(\mathcal{B}))$.

Now we define stochastic entity $T=(\mathcal{M}(T),\Delta(T))$ by setting $\mathcal{M}(T)=\mathcal{B}$ and $\Delta(T)=\text{image}({}_2\varphi)$. Further, we define $\mathcal{S}\mathcal{E}$ morphism $\varphi:=({}_1\varphi, {}_2\varphi)$. So far we have shown that ${}_1\varphi\in\text{Mor}_0(\mathcal{M}(S),\mathcal{M}(T))$ and that ${}_2\varphi$ satisfies (i).

Define map ${}_2\varphi^*$: $V(\Delta(T))^*\to V(\Delta(S))^*$ by ${}_2\varphi^*(g)(\mu)=g({}_2\varphi(\mu))$ for $g\in V(\Delta(T))^*$, $\mu\in V(\Delta(S))$. As remarked above, we can extend ${}_2\varphi$ to a linear map defined on $V(\Delta(S))$ so that ${}_2\varphi^*$ is a linear map defined on $V(\Delta(T))^*$. Further, because ${}_2\varphi[\Delta(S)]=\Delta(T)$, one establishes easily that ${}_2\varphi^*(g)\in[0,1(\Delta(S))]$ for all $g\in[0,1(\Delta(T))]$.

To establish (ii), suppose $g\in[0,1(\Delta(T))]$. If ${}_2\varphi^*(g)\notin D$, then $\exists y\in\bigcup(\mathcal{M}(S))$ with ${}_2\varphi^*(g)=f_y$. That is, $({}_2\varphi^*(g))(\mu)=\mu(y)$ for all $\mu\in\Delta(S)$. So if $\nu={}_2\varphi(\mu)\in\Delta(T)$, then $g(\nu)={}_2\varphi^*(g)(\mu)=\mu(y)={}_2\varphi(\mu)({}_1\varphi(y))=\nu({}_1\varphi(y))$. Thus, setting $x={}_1\varphi(y)$, we have that $g=g_x$. In the case ${}_2\varphi^*(g)\in D$, let $f={}_2\varphi^*(g)$ and consider $x=x_f\in\bigcup(\mathcal{M}(T))$. Then for $\nu={}_2\varphi(\mu)\in\Delta(T)$, $g(\nu)={}_2\varphi^*(g)(\mu)=f(\mu)={}_2\varphi(\mu)(x)=\nu(x)$. So again, $g=g_x$. ■

We shall use the following notation in our next theorem. Let S be a stochastic entity and $\mathfrak{G}$ a generic subset of the finite elaboration system $\mathfrak{P}=\mathfrak{P}(S)$. Suppose $\langle Y, H\rangle\in\mathfrak{G}$, $T\in Y$, and $q\in\Pi(\mathcal{M}(T))$. We denote by $f_q\ddagger$ the element $f_{q\ddagger}\in[0,1(\Delta(\mathcal{S}(\mathfrak{G})))]$. Then for $\mu\ddagger\in\Delta(\mathcal{S}(\mathfrak{G}))$, $f_q\ddagger(\mu\ddagger)=\mu\ddagger(q\ddagger)$. Note that $f_q\ddagger$ is defined for each q, but that there is no definition of $f\ddagger$ for arbitrary $f\in[0,1(\Delta(T))]$. Finally, for each $f\in[0,1(\Delta(\mathcal{S}(\mathfrak{G})))]$, we denote by $f|\Delta(T)$ the restriction of f to $\{\mu\ddagger\in\Delta(\mathcal{S}(\mathfrak{G}))|\mu\in\Delta(T)\}$.

4.2. Theorem. Let S_0 be a stochastic entity, $\mathfrak{P}$ its finite elaboration system, and $\mathfrak{G}$ a generic subset of $\mathfrak{P}$. Let $S = \mathcal{S}(\mathfrak{G})$ be the ultimate stochastic entity based on S_0 and obtained from $\mathfrak{G}$. Suppose $f \in [0, 1(\Delta(S))]$ and $T \in X(\mathfrak{G})$. Then there exists $R \in X(\mathfrak{G})$ and $x \in \bigcup(\mathcal{M}(R))$ such that (i) $T \leq R$ and (ii) $f|\Delta(R) = f_x \ddagger|\Delta(R)$.

Proof. Let $f \in [0, 1(\Delta(S))]$ and $T \in X(\mathfrak{G})$. Then there exists $\langle Y_0, H_0\rangle \in \mathfrak{G}$ with $T \in Y_0$. Define $\mathfrak{D} = \{\langle Y, H\rangle \in \mathfrak{P}|$ either (i) $\nexists\langle Y', H'\rangle$ with $\langle Y', H'\rangle \geq \langle Y, H\rangle$, $\langle Y_0, H_0\rangle$ or (ii) $\langle Y_0, H_0\rangle \leq \langle Y, H\rangle$ and $\exists R \in Y$, $\varphi \in H \cap \mathcal{S}\mathcal{E}\text{-Mor}(T, R)$ such that for every $f \in [0, 1(\Delta(R))]$, $\exists x \in \bigcup(\mathcal{M}(R))$ satisfying $f = f_x\}$.

We shall show that $\mathfrak{D}$ is dense in $\mathfrak{P}$. Consider $\langle Y, H\rangle \in \mathfrak{P}$. If $\nexists\langle Y', H'\rangle$ with $\langle Y', H'\rangle \geq \langle Y, H\rangle, \langle Y_0, H_0\rangle$, then $\langle Y, H\rangle \in \mathfrak{D}$. So if $\langle Y, H\rangle \notin \mathfrak{D}$, consider $\langle Y', H'\rangle \geq \langle Y, H\rangle$, $\langle Y_0, H_0\rangle$. Since Y' is finite, it has an element $R_0 \in Y'$ which is maximal with respect to the order relation on $\langle Y', H'\rangle$. By Lemma 4.1, there exists stochastic entity R and morphism $\varphi \in \mathcal{S}\mathcal{E}\text{-Mor}(R_0, R)$ such that ${}_2\varphi: \Delta(R_0) \to \Delta(R)$ is a bijection, and for every $g \in [0, 1(\Delta(R))$, $\exists x \in \bigcup(\mathcal{M}(R))$ with $g = g_x$. Let $Y'' = Y \cup \{R\}$ and H'' be the closure of $H \cup \{\varphi\}$ under composition. Since R_0 is a maximal in Y', we have that $\langle Y'', H''\rangle \in \mathfrak{P}$. Further, $\langle Y'', H''\rangle \geq \langle Y', H'\rangle \geq \langle Y_0, H_0\rangle, \langle Y, H\rangle$. From this we conclude that $\langle Y, H\rangle \leq \langle Y'', H''\rangle \in \mathfrak{D}$, and thus that $\mathfrak{D}$ is dense in $\mathfrak{P}$.

Because $\mathfrak{G}$ is generic, we have that there exists $\langle Y, H\rangle \in \mathfrak{G} \cap \mathfrak{D}$. Since $\mathfrak{G}$ is directed, there exists $\langle Y', H'\rangle \in \mathfrak{G}$ with $\langle Y', H'\rangle \geq \langle Y, H\rangle, \langle Y_0, H_0\rangle$. Thus, $\langle Y, H\rangle$ does not satisfy property (i) in the definition of $\mathfrak{D}$, so it must satisfy property (ii). Thus, there exists $R \in Y \subset X(\mathfrak{G})$ and $\varphi \in H \cap \mathcal{S}\mathcal{E}\text{-Mor}(T, R) \subset F(\mathfrak{G}) \cap \mathcal{S}\mathcal{E}\text{-Mor}(T, R)$ such that for every $g \in [0, 1(\Delta(R))]$, $\exists x \in \bigcup(\mathcal{M}(R))$ with $g = g_x$. Then since $f|\Delta(R) \in [0, 1(\Delta(R))]$, we have that $\exists x \in \bigcup(\mathcal{M}(R))$ with $f|\Delta(R) = f_x \ddagger|\Delta(R)$. ■

This theorem establishes a connection between general "effects" in the sense of Ludwig (1964), and effects based on single outcomes. In ultimate stochastic entities, every effect is "locally" an effect based on a single outcome. Further, if T is any stochastic entity embodying a stage of knowledge of a system, then every effect in an ultimate stochastic entity based on T is locally an effect based on an outcome in a stochastic entity R that is an *elaboration* of T. Put this way, it becomes clear how generic sets describe an order of increasing knowledge about a system.

It is useful to interpret the results above in terms of "counters." Rüttimann (1984) points out that in a general stochastic entity $(\mathcal{A}, \Delta)$, every "counter" $f \in [0, 1(\Delta)]$ may be approximated, in the weak* topology of $V(\Delta)^*$, by linear combinations of "propositional counters," f_q, $q \in \Pi(\mathcal{A})$. So for $\mu \in \Delta$, "one is tempted to interpret the value $f(\mu)$ as the long run relative frequency with which the counter f is triggered for a system in state μ." The temptation is very strong, of course, for counters f_x, $x \in \bigcup \mathscr{A}$. Hence the interpretational connection is made between the "bare-bones" structure of the stochastic entity $(\mathscr{A}, \Delta)$ and the powerful geometric structure of $V(\Delta)^*$ important in Mielnik's approach to quantum mechanics (Mielnik, 1968). The thrust of Theorem 4.2 above in this framework, therefore, is that all counters are locally propositional counters when one is considering an ultimate stochastic entity.

We shall now provide a brief introduction to "observables." More detailed explanations appear in Cohen and Rüttimann (to appear) and Rüttimann (to appear). Let S be a stochastic entity.

4.3. Definition. A *Varadarajan observable* to $\Pi(\mathscr{M}(S))$ is a map $\mathscr{o}$ from the class $B(R)$ of Borel sets of reals to $\Pi(\mathscr{M}(S))$ which satisfies the following:

(i) $\mathscr{o}(\phi) = 0$ and $\mathscr{o}(R) = 1$.

(ii) $U_1, U_2 \in B(R)$ and $U_1 \cap U_2 = \phi \Rightarrow \mathscr{o}(U_1) \perp \mathscr{o}(U_2)$.

(iii) $\{U_i\}$ is a pairwise disjoint sequence of members of $B(R) \Rightarrow \mathscr{o}(\bigcup_i U_i) = \sup\{\mathscr{o}(U_i) | i \text{ is a natural number}\}$.

4.4. Definition. The *spectrum* of a Varadarajan observable $\mathscr{o}$ is the set

$$s(\mathscr{o}) := \bigcap \{C \in B(R) | C \text{ is closed and } \mathscr{o}(C) = 1.\}$$

If $s(\mathscr{o})$ is a bounded subset of the reals, we say that $\mathscr{o}$ is a *bounded observable.*

Let $\mathscr{o}$ be a bounded observable with $a = \text{glb } s(\mathscr{o})$ and $b = \text{lub } s(\mathscr{o})$. The map $\mu \in \Delta(S) \to \int_a^b \text{Id } d(\mu \circ \mathscr{o})$ (Id is the identity map on the reals) is affine and bounded and so has an unique extension to a bounded linear functional on $V(\Delta(S))$, which we call the *expectational functional of* $\mathscr{o}$ *on* $\Delta(S)$ and denote by $\text{ex}(\Delta(S), \mathscr{o})$.

This brings us to a "local spectral theorem" for ultimate stochastic entities.

4.5. Corollary. Let S_0 be a stochastic entity, $\mathfrak{P}$ its finite elaboration system, and $\mathfrak{G}$ a generic subset of $\mathfrak{P}$. Let $S = S(\mathfrak{G})$ be the ultimate stochastic entity based on S_0 and obtained from $\mathfrak{G}$. Then for any $T \in X(\mathfrak{G})$ and any $f \in [0, 1(\Delta(S))]$, there exists $R \in X(\mathfrak{G})$ with $T \leq R$, and a bounded Varadarajan observable $\mathscr{o}$ to $\Pi(\mathscr{M}(R))$ with $f | \Delta(R) = \text{ex}(\Delta(R), \mathscr{o})$.

Proof. Given T and f, we know from Theorem 4.2 that there is an $R \in X(\mathfrak{G})$ and $x \in \bigcup \mathscr{M}(R)$ with $T \leq R$ and $f | \Delta(R) = f_x \ddagger | \Delta(R)$. For each

Borel set B in the reals, define

$$\wp(B)=\begin{cases} p(x)' & \text{if } 1\notin B, 0\in B \\ p(x) & \text{if } 1\in B, 0\notin B \\ 0 & \text{if } B=\phi. \\ 1 & \text{if } 0, 1\in B \end{cases}$$

Then for $\mu\in\Delta(R)$, $(f|\Delta(R))(\mu)=\mu(x)=\int \mathrm{Id}\ d(\mu\circ\wp)$. Since $\Delta(R)$ is a generating cone for $V(\Delta(R))$, it is not difficult to verify that the last two equalities hold for all $\mu\in V(\Delta(R))$. ■

Many results have been obtained for entities satisfying spectral theorems. See, for example, Cohen and Rüttimann (to appear), Fischer and Rüttimann (1978b), and Rüttimann (1984). We shall see in the next section that these results will be available to us for many ultimate stochastic entities.

5. STATE–STABLE SYSTEMS

In this section we consider inductive systems in which elaborations do not introduce new states. Suppose S is a stochastic entity and it is believed that $\Delta(S)$ contains all the states possible for the system under investigation by S. By that we mean that if T is an elaboration of S with $\varphi\in\mathcal{SE}\text{-Mor}(S, T)$ then ${}_2\varphi$: $\Delta(S)\to\Delta(T)$ is a surjection. Thus, every state in $\Delta(T)$ is an extension of a state in $\Delta(S)$. As we mentioned earlier, this condition reflects the situation in which the set of states of a system does not increase just because we find new experiments to test for those states.

5.1. Definition. Let S be a stochastic entity. Define $\mathfrak{P}^*$ in the same manner as we defined the finite elaboration $\mathfrak{P}$ with the stipulation that for any $\varphi\in H$, $\langle Y, H\rangle\in\mathfrak{P}^*$, ${}_2\varphi$ is a surjection. A set f generic over $\mathfrak{P}^*$ is called *state-stable.*

We shall show that for ultimate stochastic entities obtained from state-stable generic sets, the local properties established in Section 4 become global properties.

5.2. Theorem. Let S_0 be a stochastic entity, $\mathfrak{P}$ its finite elaboration system, and $\mathfrak{G}$ a state-stable, generic subset of $\mathfrak{P}$. Let $S=S(\mathfrak{G})$ be the ultimate stochastic entity based on S_0 and obtained from $\mathfrak{G}$. If $f\in[0, 1(\Delta(S))]$, then there exists an $R\in X(\mathfrak{G})$ and $x\in\bigcup\mathcal{M}(R)$ such that $f=f_x\ddagger$.

Proof. Given f, we know from Theorem 4.2 that there exists an $R\in X(\mathfrak{G})$ and $x\in\bigcup\mathcal{M}(R)$ with $f|\Delta(R)=f_x\ddagger|\Delta(R)$.

Consider any $\nu \in \Delta(S)$. Then $\exists T_1 \in X(\mathfrak{G})$ and $\mu \in \Delta(T_1)$ with $\mu\ddagger = \nu$. Since $X(\mathfrak{G})$ is directed, $\exists T_2 \in X(\mathfrak{G})$ with R, $T_1 \leq T_2$. Since $\mathfrak{G}$ is state-stable, $\exists \mu_1 \in \Delta(R)$ with ${}_2\varphi(R, T_2)(\mu_1) = {}_2\varphi(T_1, T_2)(\mu)$. Then $\mu_1\ddagger = \mu\ddagger = \nu$. Hence $f(\nu) = f(\mu_1\ddagger) = (f|\Delta(R))(\mu_1) = (f_x\ddagger|\Delta(R))(\mu_1) = (f_x\ddagger)(\mu_1\ddagger) = f_x\ddagger(\nu)$. ■

We have in ultimate stochastic entities obtained from state-stable generic sets, therefore, that every effect is based on a single outcome. This is a very powerful condition obtained in a natural way. It is a condition one would expect to hold in an "ultimate" investigation structure that requires no further elaboration.

An immediate consequence of Theorem 5.2 is a global "spectral theorem."

5.3. Corollary. Let S_0 be a stochastic entity, $\mathfrak{P}$ its finite elaboration system, and $\mathfrak{G}$ a state-stable, generic subset of $\mathfrak{P}$. Let $S = S(\mathfrak{G})$ be the ultimate stochastic entity based on S_0 and obtained from $\mathfrak{G}$. Then for every $f \in [0, 1(\Delta(S))]$, there is a Varadarajan observable $\mathscr{o}$ to $\Pi(S)$ with $f = \mathrm{ex}(\Delta(S), \mathscr{o})$. ■

This result gives an estimate of the size of the logic for an important class of ultimate stochastic entities.

Let S be an ultimate stochastic entity. Cohen and Rüttimann (to appear) define a *block* in $\Pi(\mathscr{M}(S))$ as a maximal Boolean subset. A block B is called $\Delta(S)$-dense if for every $\mu \in \Delta(S)$, μ is entirely determined by its values on B. It is shown that for a certain class of logics, the possession of a $\Delta(S)$-dense block necessarily implies that the logic is classical. (A logic is called classical if it is Boolean.) It is of interest, therefore, when considering logics that are to be essentially nonclassical, to consider those that *do not* possess $\Delta(S)$-dense blocks. The following theorem is obtained in Cohen and Rüttimann (to appear).

5.4. Theorem (Cohen, Rüttimann). Let $(P, \leq, ')$ be a nonclassical orthomodular poset with Δ a nonempty convex subset of $\Omega(P)$ such that no block in P is Δ-dense. If for every $f \in [0, 1(\Delta)]$, there is a bounded Varadarajan observable $\mathscr{o}$ to P with $f = \mathrm{ex}(\Delta, \mathscr{o})$, then P has uncountably many blocks.

We have, therefore, the following consequence of Corollary 5.3 above.

5.5. Corollary. Under the hypotheses of Corollary 5.3, if $\Pi(\mathscr{M}(S))$ is nonclassical and contains no block which is $\Delta(S)$-dense, then $\Pi(\mathscr{M}(S))$ has uncountably many blocks.

Hence ultimate stochastic entities with nonclassical logics are likely to be very large, as one might expect.

6. CONCLUSION

Randall and Foulis (1983) argue that "... the language of physics must be established on a sufficiently *general* and suitably *primitive* mathematical foundation if it is to continue to be successful as an instrument of philosophical enlightenment." It is our view that ultimate stochastic entities meet the test of generality and primitivity. It is, therefore, satisfying that they also are mathematically rich enough to exhibit the properties so useful in studies which rely on functional analysis.

In a sequel to this paper, we consider a physical entity $(\mathscr{A}, \sum)$, the basic structure in the general language developed in Foulis *et al.* (1983). We show that if $\sum$ is the set of supports of all the states in a convex set $\Delta \subset \Omega(\Pi(\mathscr{A}))$, where $(\mathscr{A}, \Delta)$ is an ultimate stochastic entity, then the cannonical map $[\cdot]: \Pi(\mathscr{A}) \to \mathscr{L}(\mathscr{A}, \sum)$ is an isomorphism. Among other things, this implies that if $\sum$ is not redundant, then every state has an "indicator outcome." In other words, to every state there corresponds an outcome that will be confirmed when tested if and only if the system is in that state. Such a plethora of outcomes might be expected in an entity embodying "ultimate" knowledge of the system. This is what happens in orthodox quantum mechanics, of course.

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